5 Working with proportionality and ratio

Key words: proportional, directly proportional, line graph, origin, gradient, slope, horizontal axis, vertical axis, *x*-axis, *y*-axis, *x*-coordinate, *y*-coordinate, rate, constant, constant of proportionality, reciprocal, inverse, inversely proportional, ratio, percentage, scale, scale drawing, scale factor, linear dimension.

There are many different kinds of relationship between variables. A very common relationship is when one variable is *proportional* to another and this section focuses on this kind of relationship. It also considers the related ideas of ratio, percentage and scale.

5.1 Meaning of proportional

A formal way of expressing proportionality would be that variable A is ***proportional*** to variable B when the values of the two variables are related by a constant multiplier. It is easier to understand the idea of proportionality through an example.

Banks count coins by weighing them. Suppose a coin has a mass of 5 g. Then two coins have a mass of 10 g. Doubling the number of coins doubles the mass. If there are 10 coins then they have 10 times the mass of one coin (i.e. 100 g). This is expressed by saying that the mass of coins is proportional to the number of coins.

A proportional relationship also works the other way round. The number of coins is proportional to the mass of the coins. A bag with 100 g of coins contains 10 coins. If you have double the mass (200 g) then you have double the number of coins (20). This is essentially what the bank is doing when it weighs coins to count them.

5.2 Proportionality and visual representation

Representing a ***proportional*** relationship as a graph can be a helpful way of exploring the idea further. The example here uses actual results of measuring the mass of a pile of 2p coins, with a reading being taken after each successive coin is added to the pile. The table of results is shown in Figure 5.1a and a ***line graph*** in Figure 5.1b (the values have been omitted from the graph for simplicity).

There are two key features of a graph showing proportionality between variables:

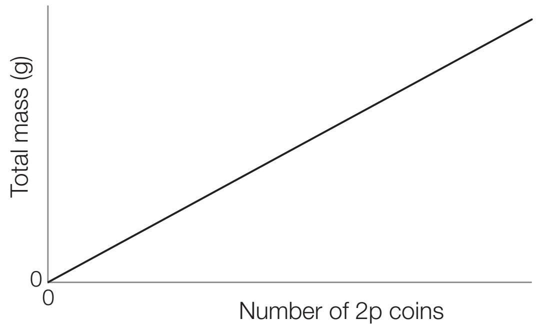
- the relationship is represented by a *straight line*
- the straight line passes through the ***origin***.

Figure 5.1 Measuring the masses of 2p coins

(a)

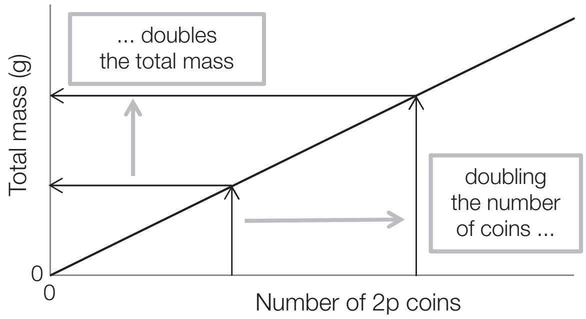
Number of 2p coins	Total mass (g)
0	0
1	7.12
2	14.24
3	21.36
4	28.48
etc.	

(b)



Doubling the numbers of coins doubles the mass (e.g. 4 coins have double the mass of 2 coins, and 6 coins have double the mass of 3 coins). Trebling the number of coins trebles the mass; halving the number of coins halves the mass. Figure 5.2 shows this idea represented on the graph. This is expressed by saying that the mass of the coins is ***proportional*** to the number of coins. (The term ***directly proportional*** is also used, but *proportional* is generally the preferred term in science. Using the term 'directly proportional' is helpful when it is being contrasted to 'inversely proportional', as explained later.)

Figure 5.2 Doubling one variable doubles the other



Note that for a relationship to be proportional, the line on the graph needs both to be straight *and* to pass through origin. A curve that passes through the origin does not represent a proportional relationship. A straight line that does not pass through the origin represents a ***linear relationship*** but not a *proportional* one. A proportional relationship is a special case of a linear relationship. (For more details, see Section 9.11 *Mathematical equations and relationships in science* on page 99.)

Note also that proportionality is not the same as *correlation* – these two terms are sometimes confused with each other. They are both concerned with the relationship between two variables but correlation applies to a different type of data. (See Section 8.7 *Relationships between variables: scatter graphs and correlation* on page 83.)

5.3 Interpretation of gradient

Figure 5.3a shows a graph with *two* lines – now representing the results for measuring the masses of two stacks of coins, of 1p as well as 2p. The line for the 2p coins is *steeper*. This implies that the mass of the 2p stack rises more than the mass of the 1p stack when a coin is added. This is because each 2p coin has a greater mass than a 1p coin. The steepness of the line is called the ***gradient*** (the term ***slope*** is also used, but *gradient* is the preferred term).

Figure 5.3 Finding the gradient of a line

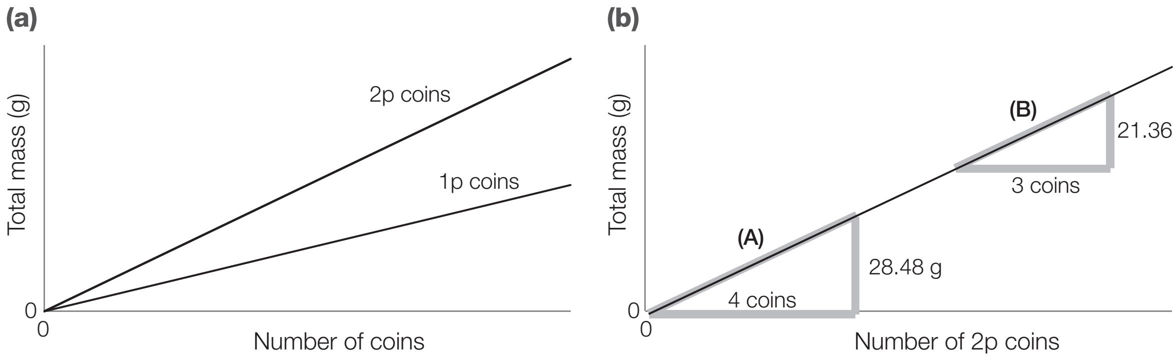


Figure 5.3b shows how the gradient of this line can be measured: it is the increase in the variable on the ***vertical axis*** or ***y-axis*** divided by the corresponding increase in the variable on the ***horizontal axis*** or ***x-axis***.

Since this is a straight line, it does not matter where this is done or how large the chosen interval is – the gradient is the same along the whole length of the line. For example, in Figure 5.3b, the gradient is measured in two different places along the line. The vertical measure is the difference between the two ***y-coordinates***, and the horizontal measure is the difference between the two ***x-coordinates***.

$$\text{gradient at (A)} = \frac{28.48\text{ g}}{4\text{ coins}} = 7.12\text{ g per coin}$$

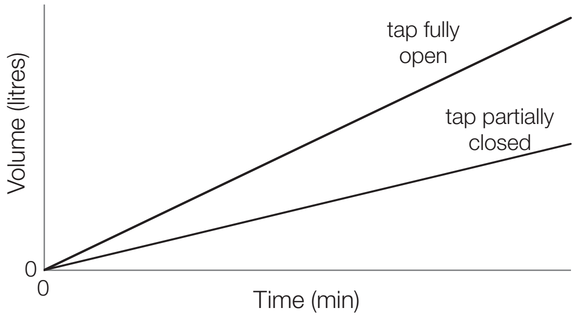
$$\text{gradient at (B)} = \frac{21.36\text{ g}}{3\text{ coins}} = 7.12\text{ g per coin}$$

The gradient works out the same for both (A) and (B), and, in this example, is in fact the mass of one coin.

Note that the two small triangles drawn on this graph are intended only to illustrate that the gradient of this straight line is the same everywhere. When finding the gradient of a line on a real graph of data, the triangle used should be drawn as large as possible (see Section 9.12 *Graphs of quantities against time: gradients* on page 103).

Another example that illustrates the meaning of a gradient is shown in Figure 5.4. The graph shows the change in the volume of water in a bath over time. At the start, the bath is empty. One line represents a fully open tap and the other a tap that is partially closed.

Figure 5.4 A bath filling with water



Since this is a change over time, the gradient represents a ***rate*** of change. In this case, the gradient is the *flow rate* of the water and is measured in litres/min. Both lines are straight and pass through the origin – this is a proportional relationship. For each tap setting, doubling the time will double the volume since the *rate* (i.e. the *gradient*) is constant. The difference in the gradients of the two lines shows that the *rate* of change is greater when the tap is fully open than when it is partially closed.

5.4 Proportionality and algebraic representation

A ***proportional*** relationship can be represented algebraically as:

$$\text{variable A} \propto \text{variable B}$$

$$\text{e.g. mass of coins} \propto \text{number of coins}$$

The symbol '$\propto$' stands for 'is proportional to'. This relationship can be expressed as a formula:

$$\text{variable A} = \text{constant} \times \text{variable B}$$

$$\text{e.g. mass of coins} = \text{mass of one coin} \times \text{number of coins}$$

The ***constant*** in the formula (in this case, 'mass of one coin') is equal to the gradient of the line on the graph. It is called the ***constant of proportionality***. The formula has the general form of the mathematical equation:

$$y = kx$$

This represents a straight line passing through the origin with a gradient of k.

Any proportional relationship 'works both ways': so if 'y is proportional to x' then it is also true to say that 'x is proportional to y'. In mathematics, this idea is expressed by saying that 'x and y are in direct proportion'.

A related kind of relationship is when one variable is proportional to the ***reciprocal*** or ***inverse*** of another variable, i.e.

$$y \propto \frac{1}{x}$$

This would be described by saying that y is ***inversely proportional*** to x: if x is doubled then y is halved. The general form of the equation would be:

$$y = \frac{\text{constant}}{x}$$

These ideas about *directly proportional* and *indirectly proportional* relationships are illustrated with some common examples from school science in the next section.

5.5 Proportional relationships in science

Some ***proportional*** relationships in science arise from *definitions* of quantities and others are derived from *experimental observations*. An example of a definition is:

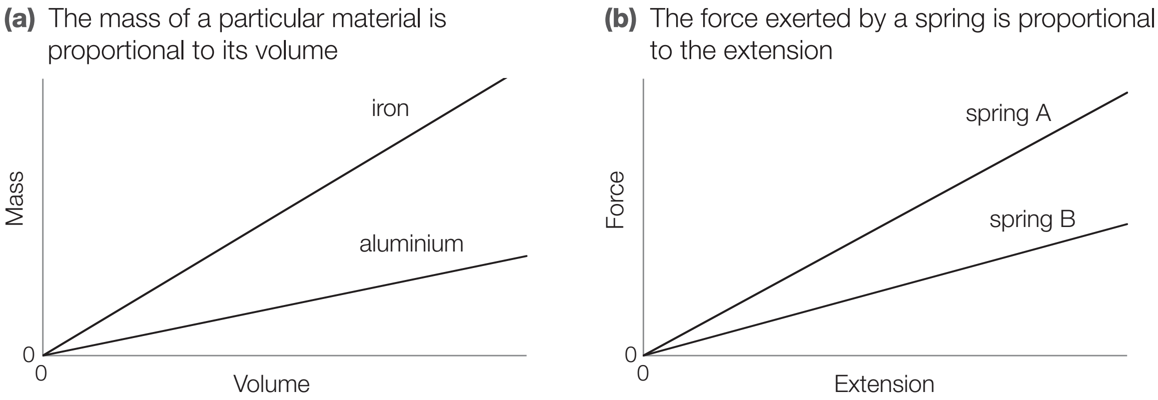
$$\text{density} = \frac{\text{mass}}{\text{volume}}$$

This is not of the form $y = kx$ but it can be rearranged. (For details on rearranging formulae, see Chapter 9 *Scientific models and mathematical equations* on page 87.)

$$\text{mass} = \text{density} \times \text{volume}$$

For objects made of the same material (i.e. constant density), the mass is proportional to the volume. In Figure 5.5a, the gradient for 'iron' is greater than for 'aluminium' because iron is a more dense material. Here, the ***constant of proportionality*** is the density of the material and this can be found by calculating the gradient of the straight line.

Figure 5.5 Proportional relationships



An example of a relationship derived from experiment is Hooke's Law. By experiment, it was found that the force exerted by a spring is proportional to the extension (within the elastic limit of the spring). The constant of proportionality is called the spring constant and it is a characteristic of a particular spring: the larger the spring constant, the stiffer the spring (Figure 5.5b).

$$\text{force exerted} = \text{spring constant} \times \text{extension}$$

For each relationship in science, there tends to be a 'conventional' way of expressing the ***formula***, and this may not always have the form $y = kx$. The formula may need to be rearranged to express it in this way. In addition, what is considered to be the variable and what is considered to be the constant depends on the context. For example, if bottles of the same fixed volume are each filled with liquids of different densities, the *mass* of the liquid is proportional to the *density* and the *volume* would be the constant.

Figure 5.6 shows selected relationships, with the constants in the formulae underlined.

Figure 5.6 Examples of constants in proportional relationships

Relationship	Formula (constant is underlined)
For an object made of a particular material: mass ∝ volume	mass = volume × <u>density</u>
For filling a fixed volume with different liquids: mass ∝ density	mass = <u>volume</u> × density
For a car travelling at constant speed along a motorway: distance travelled ∝ time	distance travelled = <u>speed</u> × time
For an object moved by a constant force: work done ∝ distance	work done = <u>force</u> × distance
For a resistor that obeys Ohm's Law (i.e. a constant resistance): potential difference ∝ current	potential difference = current × <u>resistance</u>

Note that saying that something is a *constant* does not mean that it is just a number with no units. In the algebraic equation $y = kx$, the constant of proportionality k does represent just a number. However, in the examples in the above table, all of the constants are values with *units*. Thus, for the first example, mass (g) is proportional to volume (cm^3), and the constant is density (g/cm^3).

Rearranging the kinds of formulae shown in the table above can reveal relationships that are ***inversely proportional***. For example:

$$\text{wave speed} = \text{frequency} \times \text{wavelength}$$

For light in a particular medium, the wave speed is constant. Rearranging the formula brings out more clearly the relationship that frequency is inversely proportional to wavelength.

$$\text{frequency} = \frac{\text{wave speed}}{\text{wavelength}}$$

This means that if the wavelength is *doubled* then the frequency is *halved*. If it is trebled (or multiplied by any amount k) then the frequency is divided by three (or divided by the amount k).

For more details about directly proportional and inversely proportional relationships, see Section 9.11 *Mathematical equations and relationships in science* on page 99.

5.6 Ratios

A ***ratio*** is a comparison of two *similar quantities* and thus does not have units. For example, the mass of a 1p coin is 3.56 g and the mass of a 2p coin is 7.12 g. Thus the ratio of the mass of a 1p coin to the mass of a 2p coin is 3.56 : 7.12 (no units). This reduces to 1 : 2. The mass of a 2p coin is exactly twice that of a 1p coin and so, in this example, the ratio consists of *integers* (whole numbers).

Similarly, in aluminium oxide (Al_2O_3), the ratio of aluminium atoms to oxygen atoms is 2 : 3 – again integers. It is also possible to express this as 1 : 1.5. Which of these ways of expressing a ratio is better is a matter of choice, depending on what is more useful for the context.

The ratio of the width of a sheet of A4 paper (210 mm) to the height (297 mm) is 210 : 297. This is a rather unwieldy ratio. In such cases, the ratio is expressed in the form ‘1 : x’. For A4 paper, this would be 1 : 1.414. Using a ratio in this form makes comparisons with other ratios easier. For example, the ratio of the width to the height for A3 paper is the same (1 : 1.414) as for A4, showing that the two sizes of paper are similar shapes.

In some ratios, the two quantities being compared are also *parts* of a *whole*. For example, the ratio of aluminium atoms to oxygen atoms in Al_2O_3 is 2 : 3, and here it is meaningful to add the ‘2’ and ‘3’ together to give ‘5’, since this represents the *total number of atoms* in the formula. Thus, ⅖ of the atoms in aluminium oxide are oxygen atoms (or 0.4 or 40%).

5.7 Proportional reasoning and ratios

The following is an example of a calculation that appears to be relatively straightforward: 2 cm^3 of aluminium has a mass of 5.4 g. What is the mass of 4 cm^3? (Answer: 10.8 g)

The simplest method of arriving at the answer is to reason that doubling the volume (from 2 cm^3 to 4 cm^3) will double the mass (from 5.4 g to 10.8 g). Although this seems intuitive, it does in fact involve a rather subtle idea – in effect, *comparing two ratios* to find x:

$$\text{volume 1 (cm}^3) : \text{volume 2 (cm}^3) = \text{mass 1 (g)} : \text{mass 2 (g)}$$

$$2 : 4 = 5.4 : x$$

Finding the value of x from these ratios involves ***proportional*** reasoning. While using these simple ratios might not be too difficult, it becomes conceptually harder when the mental manipulation of the values is more challenging. For example, 17 cm^3 of aluminium has a mass of 91.8 g. What is the mass of 63 cm^3?

In such a case, it may be easier to do this as a two-stage calculation, working out first the density of aluminium (i.e. the mass of 1 cm^3). This value can then be used to calculate the mass of 63 cm^3 of aluminium.

For further details about different calculation strategies, see Chapter 9 *Scientific models and mathematical equations* on page 87.

5.8 Percentages

A ***percentage*** is a kind of fraction that relates a *part* to a *whole*. Using a percentage is helpful when comparing one thing to another, because it can avoid unwieldy fractions or decimals.

For example, if a population of 200 rabbits (the *whole*) has 60 males (the *part*) then the proportion of males in the population can be expressed in any of the following ways:

$$\frac{60}{200} \qquad \frac{3}{10} \qquad \frac{30}{100} \qquad 30\%$$

Thus, if the proportion of the part to the whole is expressed as a fraction with 100% as the denominator then the percentage is the numerator:

$$\frac{\text{part}}{\text{whole}} = \frac{\text{percentage}}{100\%}$$

This equation can be rearranged so that any one of these values (part, whole or percentage) can be calculated from values for the other two (for details of rearranging equations, see Chapter 9 *Scientific models and mathematical equations* on page 87). For example, a percentage can be calculated from:

$$\text{percentage} = \frac{\text{part}}{\text{whole}} \times 100\%$$

However, difficulties in calculations involving percentages can arise because of confusion over what the 'part' and the 'whole' represent. Avoiding the inappropriate use of formulae requires an understanding of what the percentage means in the context of the problem. For example, a percentage may apply to a part of an existing whole, or to an increase, or to a decrease.

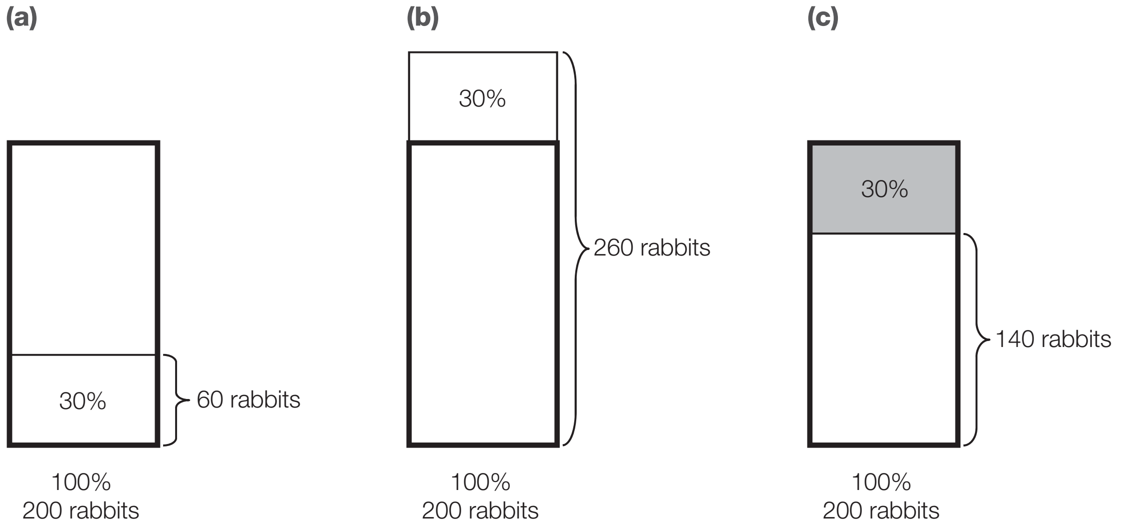
The following questions are represented visually in Figure 5.7, which emphasises the meaning of the percentage in each case.

(a) A population of 200 rabbits has 30% males. How many males are there? (Answer: 60 male rabbits)

(b) A population of 200 rabbits increases by 30%. How big is the population after the change? (Answer: 260 rabbits)

(c) A population of 200 rabbits decreases by 30%. How big is the population after the change? (Answer: 140 rabbits)

Figure 5.7 Different meanings of a percentage



Pupils can get a better feeling for the idea that a percentage represents a fraction (a part of a whole) if they are familiar with some common examples: 50% represents ½, 25% represents ¼, 20% represents ⅕, and so on.

Note that, although percentages can be effective for communicating values, they may not always be the most useful form for doing calculations. For example, saying that something has a 10% chance of happening is the same as saying that it has a probability of 0.1; the former communicates a clear message but the latter is more convenient for use in calculations on probabilities.

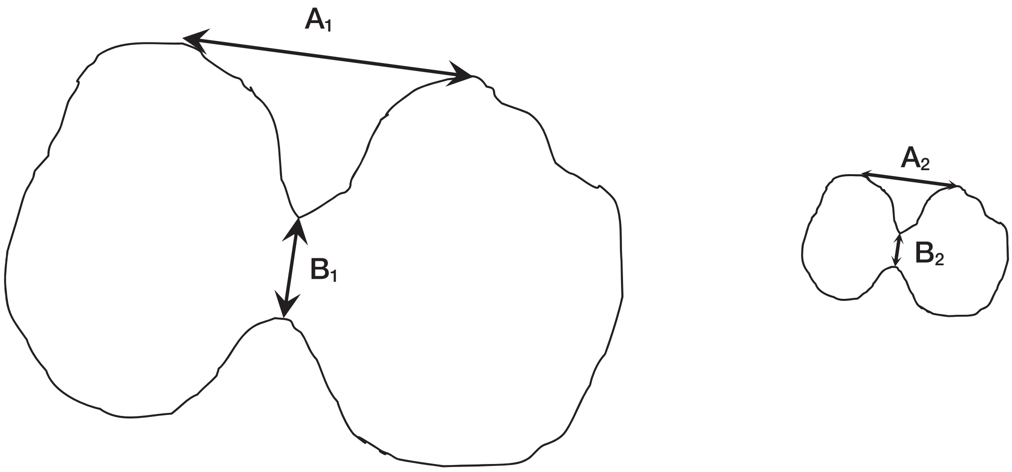
5.9 Scale drawings and images

To ***scale*** a quantity means to enlarge or reduce it by a given amount. A ***scale drawing*** of an object is one in which all of the dimensions of the original object are multiplied by a constant. This constant is called the ***scale factor*** (another example of a ***constant of proportionality***). Pupils encounter scale drawings in biology (e.g. images of microscopic organisms) and in physics (e.g. representations of forces).

If the scale factor is *greater than 1* then this produces an *enlarged* image (e.g. a drawing of a bacterium). If the value of the scale factor is *between 0 and 1* then this produces a *reduced* image (e.g. a map).

In the example shown in Figure 5.8, the original on the left has been *reduced* 3 times to produce the scale drawing on the right, i.e. the scale factor is ⅓. Every measurement is scaled by the same factor, so A_2 is ⅓ times A_1 , and B_2 is ⅓ times B_1.

Figure 5.8 Original image and scale drawing



Another way of representing the scale factor would be to say that the scale of the drawing is 1 : 3. Note that, in this ratio, the first number represents the dimension of the scale drawing and the second number represents the dimension of the original. Other examples of scales as ratios would be a model aeroplane with a scale of 1 : 72 and a map with a scale of 1 : 50 000.

For drawings and photographs of microscopic objects, where the image is enlarged, the scale factor is usually represented as a magnification. For example, '100×' may appear next to an image meaning that it is 100 times larger than the original (i.e. the scale factor is 100). Interpreting such images requires an understanding both of scaling and of the units used to describe the sizes of microscopic objects (see Section 2.6 *Dealing with very large and very small values* on page 20).

Note that the scale factor applies only to the ***linear dimensions***. For the effects of scaling on areas and volumes, see Chapter 10 *Mathematics and the real world* on page 107.